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Decoherence of a central quantum system coupled to an XY spin chain

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Abstract

We investigate decoherence of a central quantum system uniformly coupled to an XY spin-$1/2$ bath in a transverse field. Through explicitly calculating the Loschmidt echo (LE) used to characterize decoherence quantitatively we find that the anisotropy parameter γ sensitively affects the decoherence of the central system when $\gamma \in [0, 1]$. Interestingly, the LE becomes unit under the condition that the initial state of the environment is a product state. Although it is difficult to control the environment to be product states in reality, our findings may provide a new understanding of mechanism of the decoherence.

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1. Introduction

Coherence of a quantum state is very fragile because of the existence of its environmental degrees of freedom coupled to it, which has become the major obstacle in constructing quantum computer [1, 2]. To protect the quantum information, we generally use the quantum error correction scheme which can correct the quantum errors to protect the encoded quantum states [3–5]. We can also use the scheme to find the decoherence-free subspaces and some other schemes to protect the quantum information [6, 7]. Generally we need several physical qubits to realize one logic qubit in these schemes. It will be very interesting if we can find a quantum systems in which the quantum states can be naturally protected.

On the other hand, many physicists took attention to the relationship among the concepts of environment, decoherence and irreversibility; these investigations may provide new perspective of how to overcome decoherence and renewed understanding for the crossover between quantum and classical behaviour [8]. In the study of quantum-classical transition in quantum chaos, the concept of Loschmidt echo (LE) [9] was introduced, we also employ it to characterize the decoherence of a central system. With the development of quantum information, entanglement was used to investigate the quantum phase transition (QPT) [11, 12]. Very recently Quan *et al* have found that the quantum critical behaviour of the environmental system strongly affects its capability of enhancing the decay of the LE [13]. In this paper, we present a theoretical study of the behavior of the LE characterizing the decoherence quantitatively by extending the Ising mode used in [13] to a more general XY model, and find that the anisotropy parameter γ sensitively affects the decoherence of the central system when $\gamma \in [0, 1]$. Interestingly, the LE becomes unit under the condition that the initial state of the environment is a product state.

2. Derivation of the LE for a central system

Firstly, we analyse the XY model as a starting point, since it is exactly solvable and presents a rich structure. The system-bath model can be described by the Hamiltonian $\mathcal{H} = \mathcal{H}_S + \mathcal{H}_E + \mathcal{H}_{\rm SE}$, the central (two-state) system, characterized by the ground state $|0\rangle$ and the excited state $|1\rangle$, has a free Hamiltonian $\mathcal{H}_S = w_e|1\rangle\langle 1|$ and is coupled to all spins in the bath through the interaction $\mathcal{H}_{\rm SE} = -\delta \sum_l |1\rangle\langle 1|\sigma_l^z$, where δ represents the coupling constant. Our model differs from the model in [14] where the system only interacts with the first spin in the bath. The chain in a transverse field has nearest-neighbour interactions with Hamiltonian expressed by

$$\mathcal{H}_E = -\sum_{l=-M}^{M} \left(\frac{1+\gamma}{2} \sigma_l^x \sigma_{l+1}^x + \frac{1-\gamma}{2} \sigma_l^y \sigma_{l+1}^y + \lambda \sigma_l^z \right), \tag{1}$$

where $M = (N-1)/2$ for N odd, and the operators σ_l^α $(\alpha = x, y, z)$ are the usual Pauli operators defined on the lth site of the lattice. The constants $\gamma \in [0, +\infty]$ and $\lambda \in \mathrm{R}$ represent the anisotropy parameter in the next-neighbour spin–spin interaction and an external magnetic field. The model defined by equation (1) has a rich structure [15], i.e., when the anisotropy parameter γ is set to $(0, 1]$, the model of equation (1) belongs to the Ising universality class which has a critical point only at $\lambda_c = 1$; however, when $\gamma = 0$, it belongs to the XY universality class and the critical region is $\lambda_c \in (-1, 1)$.

We assume the central system to be prepared in a superposition state $|\Psi_S\rangle = \alpha|0\rangle + \beta|1\rangle$, thus the initial system-environment state can be written as $\Psi_{\rm SE}(0)\rangle = (\alpha|0\rangle + \beta|1\rangle)|\Psi_E(0)\rangle$. From the evolved reduced density matrix of the system $\rho_S(t) = \mathrm{Tr}_E|\Psi_{\rm SE}(t)\rangle\langle\Psi_{\rm SE}(t)|$, we obtain

$$\rho_S(t) = |\alpha|^2|0\rangle\langle 0| + \alpha\beta^* R(t)|0\rangle\langle 1| + \alpha^*\beta R^*(t)|1\rangle\langle 0| + \beta^2|1\rangle\langle 1|. \tag{2}$$

Clearly on the basis of the eigenstates $|0\rangle$ and $|1\rangle$, the diagonal terms in equation (2) do not evolve with time, and only the off-diagonal terms will be modulated by the decoherence factor $R(t)$, which is the overlap between two states of the environment obtained by evolving the same initial state $|\Psi_E(0)\rangle$ driven by two different effective Hamiltonians $\mathcal{H}_0$ and $\mathcal{H}_1$. As discussed in [13], for the model (1) we have $\mathcal{H}_j = -\sum_{l=-M}^{M} \left[\frac{1+\gamma}{2}\sigma_l^x\sigma_{l+1}^x + \frac{1-\gamma}{2}\sigma_l^y\sigma_{l+1}^y + (\lambda + j\delta)\sigma_l^z\right]$ with $j = 0$ or 1. $R(t)$ can be defined as

$$R(t) = \langle\Psi_E(0)|\, \mathrm{e}^{\mathrm{i}t\mathcal{H}_0}\, \mathrm{e}^{-\mathrm{i}t\mathcal{H}_1} |\Psi_E(0)\rangle, \tag{3}$$

while the LE is determined from $L(t) = |R(t)|^2$, which is also called fidelity. If the initial surrounding environment is prepared in the ground state of $\mathcal{H}_0$, i.e., $|\Phi_0\rangle$, equation (3) will reduce to a simpler form

$$R(t) = \langle\Phi_0|\, \mathrm{e}^{-\mathrm{i}t\mathcal{H}_1} |\Phi_0\rangle, \tag{4}$$

where an irrelevant phase factor is removed.

Next we will deduce the detailed expression of $R(t)$ for model (1). In the standard way, the two Hamiltonians $\mathcal{H}_j$ can be diagonalized in terms of a suitable set of fermionic creation and annihilation operators $\mu_k^{(j)}$ as

$$\mathcal{H}_j = \sum_{k=1}^{M} \varepsilon_k^{(j)} \left[\mu_k^{(j)\dagger} \mu_k^{(j)} - 1\right]. \tag{5}$$

When getting the equation above, we have applied to each spin a rotation of ϕ around the z direction $\mathcal{H}_j(\phi) = g(\phi)\mathcal{H}_j g^{\dagger}(\phi)$ with $g(\phi) = \prod_{l=-M}^{M} \exp\left(\mathrm{i}\sigma_l^z \phi/2\right)$, the Jordan–Wigner transformation mapping the spins to one-dimensional spinless fermions with creation and annihilation operators a_l and $a_l^{\dagger}$ via the relation $a_l = \left(\prod_{i<l} \sigma_i^z\right)\left(\sigma_l^x + \mathrm{i}\sigma_l^y\right)/2$, and the Fourier transformation of the fermionic operators described by $c_k = (1/\sqrt{N}) \sum_l a_l \exp(-\mathrm{i}2\pi lk/N)$. The energy spectrum in equation (5) is

$$\varepsilon_k^{(j)} = \sqrt{\left[\cos\left(\frac{2\pi k}{N}\right) - (\lambda + j\delta)\right]^2 + \gamma^2 \sin^2\left(\frac{2\pi k}{N}\right)}, \tag{6}$$

and through a Bogliubov transformation the operators appearing in the Hamiltonians $\mathcal{H}_j$ we have

$$\mu_k^{(j)} = c_k \cos\left[\frac{\theta_k^{(j)}}{2}\right] - \mathrm{i}c_{-k}^{\dagger}\, \mathrm{e}^{2\mathrm{i}\phi} \sin\left[\frac{\theta_k^{(j)}}{2}\right], \tag{7}$$

where the angles $\theta_k^{(j)}$ is the Bogliubov coefficients satisfying the following equation:

$$\cos\left[\theta_k^{(j)}\right] = \frac{\cos\left(\frac{2\pi k}{N}\right) - (\lambda + j\delta)}{\varepsilon_k^{(j)}}. \tag{8}$$

It is easy to check that the spinless Fermion operators $\mu_{\pm k}^{(j)}$ satisfy

$$\mu_{\pm k}^{(0)} = \mu_{\pm k}^{(1)} \cos(\theta_k) \mp \mathrm{i}\mu_{\mp k}^{(1)\dagger}\, \mathrm{e}^{2\mathrm{i}\phi} \sin(\theta_k), \tag{9}$$

where $\theta_k = \left[\theta_k^{(0)} - \theta_k^{(1)}\right]/2$.

According to equation (9), the ground state of $\mathcal{H}_0$ can be expressed as

$$|\Phi_0\rangle_{XY} = \prod_{k=1}^{M} [\cos(\theta_k)|0\rangle_k|0\rangle_{-k} + \mathrm{i}\, \mathrm{e}^{2\mathrm{i}\phi} \sin(\theta_k)|1\rangle_k|1\rangle_{-k}], \tag{10}$$

for any operators $\mu_{\pm k}^{(0)}$ we have $\mu_{\pm k}^{(0)}|\Psi_0\rangle = 0$. $|0\rangle_k$ and $|1\rangle_k$ are the vacuum and single excitation of the kth mode, $\mu_k^{(1)}$, respectively.

As expected that the ground state of $\mathcal{H}_0$ is taken as the initial surrounding environment state, substituting equation (10) into equation (4) we obtain the decoherence factor

$$R(t) = \prod_{k=1}^{M} R_k(t) = \prod_{k=1}^{M} \left[\sin^2(\theta_k) + \cos^2(\theta_k)\, \mathrm{e}^{\mathrm{i}2\varepsilon_k^{(1)} t}\right], \tag{11}$$

so we can express the LE as

$$L(t) = |R(t)|^2 = \prod_{k=1}^{M} \left[1 - \sin^2(2\theta_k) \sin^2\left(\epsilon_k^{(1)} t\right)\right]. \tag{12}$$

The term $R_k(t) \equiv \sin^2(\theta_k) + \cos^2(\theta_k)\, \mathrm{e}^{i2\varepsilon_k^{(1)} t}$ is a decoherence factor for the kth mode, and its modulus square is always not larger than one. It is interesting to mention that the Berry

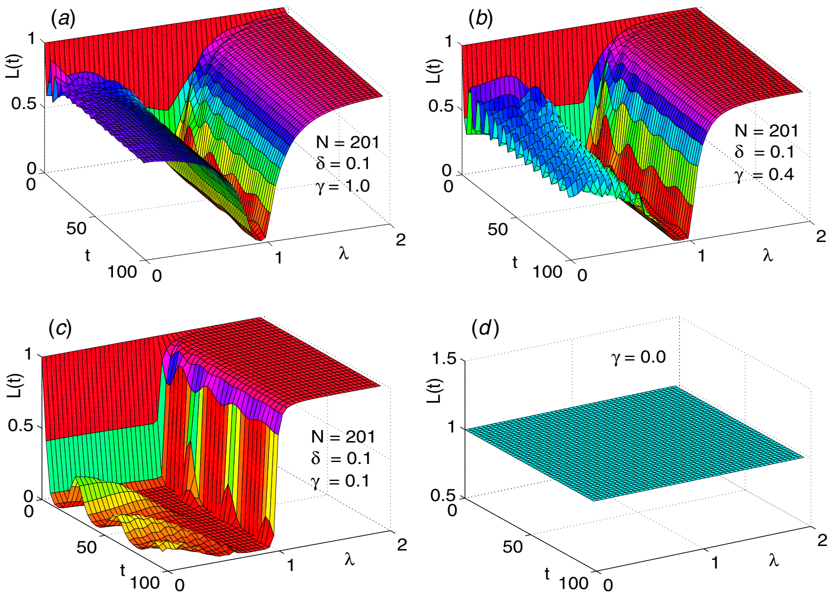

Figure 1. Three-dimensional diagram of $L(t)$ as a function of t and λ with $N = 201$ and $\delta = 0.1$. With decreasing γ from 1.0 to 0.1, (*a*)–(*c*) show that the range of λ where the decay of $L(t)$ is enhanced increases. Part (*d*) shows that when $\gamma = 0.0$, $L(t)$ is unit always, regardless of what N, λ and δ are.

(This figure is in colour only in the electronic version)

phase of the ground state in the XY model is of sum form for each mode [16, 17], while this decoherence factor (11) is of multiplying form for each mode. Furthermore, equation (11) is analogous to that for non-interacting spin environments [19, 18, 20] and Cucchietti generalized Quan's results [21].

To better understand the LE (12), we plot it as a function of t and λ in figure 1, and as a function of only t in figure 2. For simplicity, we only set $N = 201$ and $\delta = 0.1$. It is demonstrated that the decay of $L(t)$ is enhanced at the critical point of quantum phase transition $\lambda_c = 1$ in figure 1(*a*), since the XY model with $\gamma = 1$ corresponds to the Ising model. There exists a deep valley in the around the line $\lambda = 0.9$, which is the same results as [13]. However, for the general XY model where γ is adjusted in (0, 1), we find that the decay of $L(t)$ is enhanced in a different degree in the range $\lambda \in (0, 1)$. When $\gamma = 0.4$, the amplitudes of $L(t)$ in figure 1(*b*) is smaller than those corresponding to figure 1(*a*) and the range of λ resulting in $L(t) = 0$ increases. When we continue to decrease γ to 0.1 in figure 1(*c*), it is seen that $L(t)$ nearly approaches zero in the range $\lambda \in (0, 1)$, where the central system transits from a pure state to a mixed state. So we can conclude that for a smaller γ, the critical point of quantum phase transition is the transition point of whether the decay of $L(t)$ is enhanced or not.

Comparing with the results in [13], for the general XY model, we also see that $L(t)$ decays and revives as time increases in figures 2(*e*) and (*f*). This may serve as a witness of QPT. At the same time, if we appropriately adjust the parameters N, δ and λ as shown in figures 2(*e*) and (*f*), it is found that the two plots of $L(t)$ with the same γ have the identical profile, indicating that the period of the revival of $L(t)$ is proportional to the size of the surrounding system in

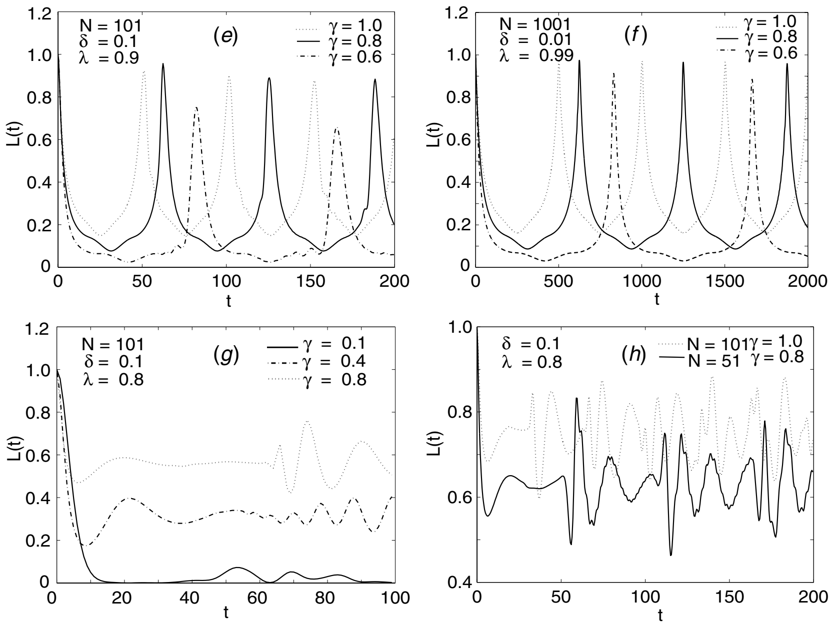

Figure 2. $L(t)$ as a function of t. Parts (e) and (f) show that with decreasing γ the quasiperiod of $L(t)$ increases, and it is proportional to the size of the surrounding system. The quasiperiod stems from quantum phase transition at $\lambda_c = \lambda + \delta = 1$. Part ($g$) shows that with decreasing γ the decay of $L(t)$ is enhanced faster. Part (h) shows that for some parameters λ (away from critical point $\lambda_c = 1$) $L(t)$ becomes chaotic, which is due to the competition between the two phases separated by $\lambda_c = 1$.

the case of finite N. Figures 2(g) and (h) reflect that the decreasing γ leads to fast decaying of the $L(t)$, which complies with the situation described by figures 1(a)–(c). In quantum chaos [9] the sensitivity of perturbations in the Hamiltonian system can be understood according to the LE [22]. Here, for some parameters shown in figures 2(g) and (h), $L(t)$ becomes chaotic, which is due to the competition between the two phases separated by $\lambda_c = 1$.

3. Special cases of the LE being unit

Interestingly, we find that $L(t)$ does not vary with time in the XX model with $\gamma = 0$, i.e., the coherence of the central spin will not be affected by the special environment. The reasons are in the following. From equations (6)–(8), we see that

$$\lim_{\gamma\to 0} \cos\left[\theta_k^{(j)}\right] = \pm 1, \tag{13}$$

which directly results in that the ground state of $\mathcal{H}_0$ no longer lies in the two-dimensional Hilbert space spanned by $|0\rangle_k|0\rangle_{-k}$ and $|1\rangle_k|1\rangle_{-k}$, but only one of them like equation (14). To obtain the explicit form of the ground state, we let $\cos(2\pi k_1/N) = \lambda$ and $\cos(2\pi k_0/N) = \lambda+\delta$ in equation (8), and from them know that $k_0 < k_1$. Considering equations (8–10), we can express the ground state

$$|\Phi_0\rangle_{XX} = \prod_{k=1}^{k_0} |0\rangle_k|0\rangle_{-k} \prod_{k=k_0+1}^{M} (\mathrm{i}e^{2\mathrm{i}\phi})|1\rangle_k|1\rangle_{-k}, \tag{14}$$

since $\cos(\theta_k) = 1$ in equation (10) when $k < k_0$, and $\mu_k^{(0)} = \mu_k^{(1)}$ $(k < k_0)$ or $-\mu_k^{(1)}$ $(k > k_0)$. Note that the state in equation (14) is a product state, which stems from $\gamma = 0$. Substituting equation (14) into equation (4), we have

$$L(t) = \lim_{\gamma \to 0} |R(t)|^2 = 1, \tag{15}$$

which means that $L(t)$, initial equal to 1, will not decay at all, and the central system preserves its initial coherence except for an additional phase factor in one of eigenstates of the central system. For our model with the central system surrounded by the XX spin-1/2 bath, note that the purity [13, 18] $P = \mathrm{Tr}_S(\rho_S^2)$ is defined to describe decoherence, it is $P = 1 - 2|\alpha\beta|^2[1 - L(t)] = 1$ also, independent of the central system and this special environment. What is more, result (15) is regardless of the number N of the lattice and what the external magnetic field λ is, it seems to be counterintuitive, but it is indeed the case. We can see in figure 1(*d*). At the same time, we emphasize that the non-decay of the LE in this case is nontrivial because of the difference between $\mathcal{H}_0$ and $\mathcal{H}_1$.

The result can be better understood as follows. Evolving from the initial system-environment state $|\Psi_{\mathrm{SE}}(0)\rangle = (\alpha|0\rangle + \beta|1\rangle)|\Psi_E(0)\rangle$, which is not entangled, it becomes $|\Psi_{\mathrm{SE}}(t)\rangle = \alpha|0\rangle|\Phi_0(t)\rangle + \beta|1\rangle|\Phi_1(t)\rangle$ at an arbitrary t, where $|\Phi_0(t)\rangle$ and $|\Phi_1(t)\rangle$ are driven by the Hamiltonians $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. It is known that $\mathcal{H}_0$ and $\mathcal{H}_1$ are different; however, it happens that the anisotropy parameter $\gamma = 0$ leads to their same evolution, resulting in $\langle\Phi_0(t)|\Phi_1(t)\rangle = \exp(\mathrm{i}\varphi)$. The real time-dependent φ denotes an additional phase factor.

Finally, we assume that the XX model environment to be initially prepared in an arbitrary excited state with $\gamma = 0$. An n-particle state has the form $\mu_{k_1}^{(0)\dagger}\mu_{k_2}^{(0)\dagger}\cdots\mu_{k_n}^{(0)\dagger}|\Phi_0\rangle_{XX}$, with all the k_i distinct, i.e., it is

$$|\Phi_n\rangle_{XX} = \prod_{k'=k_1}^{k_n} |1\rangle_{k'}|0\rangle_{-k'} \prod_{k=1}^{k_0} |0\rangle_k|0\rangle_{-k} \prod_{k=k_0+1}^{M} (\mathrm{i}e^{2\mathrm{i}\phi})|1\rangle_k|1\rangle_{-k}, \tag{16}$$

where $k \neq k'$. Substituting equation (16) into equation (4), we find that the LE is $L(t) = 1$ also, which implies that the partial excited states of the environment does not induce decoherence to the central system. However, if the environment is initially prepared in a thermal state, the LE is no longer equal to unit, but will decay with time. Of particular interest is the case in which the XY model lies initially in an excited state. The m-particle state can be written as

$$|\Phi_m\rangle_{XY} = \prod_{k'=k_1}^{k_m} |1\rangle_{k'}|0\rangle_{-k'} \prod_{k=1,k\neq k'}^{M} [\cos(\theta_k)|0\rangle_k|0\rangle_{-k} + \mathrm{i}e^{2\mathrm{i}\phi}\sin(\theta_k)|1\rangle_k|1\rangle_{-k}]. \tag{17}$$

After calculation by substituting equation (17) into equation (4) the LE is derived as $L(t) = \prod_{k=1}^{M}\left[1 - \sin^2(2\theta_k)\sin^2\left(\varepsilon_k^{(1)}t\right)\right]$ with $k \neq k'$. It can be seen that (i) the excited states $\prod_{k'=k_1}^{k_m}|1\rangle_{k'}|0\rangle_{-k'}$ have no any contributions to modulating the LE; (ii) if all the particles are excited, i.e., $|\Phi_m\rangle_{XY} = \prod_{k=1}^{M}|1\rangle_k|0\rangle_{-k}$, which is assumed to be the initial state of the bath, the LE of the central system is unit also. Note that if the initial state of the XY model bath is thermal, the LE of the central system will decay with time. The vanishing decoherence may arise in view of the non-interacting environments in [19]: if we let all spins lie initially in either up or down, the decoherence factor will be unit as well. It is worthwhile for us to find out its physical nature.

4. Conclusions

In summary, we investigate the decoherence process of a central quantum system uniformly coupled to an XY spin-1/2 bath in a transverse field. Through explicitly calculating the LE used to characterize decoherence quantitatively we find that the anisotropy parameter γ sensitively affects the decoherence of the central system when $\gamma \in [0, 1]$. Interestingly, the LE becomes unit under the condition that the initial state of the spin chain environment is a product state: the initial state of the XX spin-1/2 bath lies either in the ground state or in the state that the partial particles are excited, or in the state that all particles are excited. Although it is difficult to make the initial state of the spin chain environment be in a product state at zero temperature, in a theoretical-investigation sense, our findings may shed light on understanding of the mechanism of the decoherence.

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